Development of Nonstandard Discrete Simulation Model for the Numerical Solution of Second Order Ordinary Differential Equations

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Abstract
The paper presents the use of renormalized denominator function in the construction of suitable discrete model that can be used for numerical analysis of initial value problems arising from second order ordinary differential equations. In the paper, we used the nonstandard rule on renormalization of denominator functions to construct qualitatively suitable numerical model for some second order ordinary differential equations.
Keywords: Discrete Model, Simulation, Renormalized Denominator, Nonstandard Rule.

Introduction

The mathematical formulation of several physical phenomena results in differential equations which are usually non-linear. Researchers attempts the solution to such equations by replacing the non-linear equations with related linear equations which approximates the original equation in such a way that it produces solution that behave close enough to the dynamics of the behaviour of the original phenomena.

However, such "linearization" is not always feasible. This being the case, mathematicians result to various forms of approximations that are used to arrive at some desired result. In general, numerical models have been in the fore front of such approximations. A lot of researchers had worked on discrete models for non-linear differential equations. These include Anguelov and Lubuma (2003), Fatunla (1988), Mickens (1981), Mickens (1994) and Stretter (1973) to mention a few.

In this research work, we shall create new discrete model for the solution of some special linear and non-linear ordinary differential equations.

In particular, we shall consider the following equations

i. $y''y' - 4x$

ii. $y'' - x\,(y')^2 = 0$

iii $y'' - 2(y' - y) = 0$

Derivation of the Scheme

The general nonstandard finite difference scheme for second order ordinary differential equation can be derived by replacing the first and second order derivatives in the following manner in line with the rules of Fatunla (1988) and Mickens (1994).

$$y'' \equiv \frac{y_{k+1} - 2y_{k} + y_{k-1}}{\varphi} \qquad (1)$$

where $\varphi(h) \to h^2 + 0(h^4)\ as\ h \to 0$ $\qquad y' \equiv \frac{(y_{k+1} - y_k)}{\psi}$ (2)

where $\psi(h) \rightarrow h + 0(h^2)\ as\ h \rightarrow 0$

$y' \equiv \frac{(y_{k+1} - \beta y_k)}{\psi}$ (3)

where $\psi(h) \rightarrow h + 0(h^2),\ \beta(h) \rightarrow 1\ as\ h \rightarrow 0$

$y' \equiv \frac{(y_k - \beta y_{k-1})}{2\psi}$ (4)

where $\psi(h) \rightarrow h + 0(h^2),\ \beta(h) \rightarrow 1\ as\ h \rightarrow 0$

Suitable functions for the above include

$\varphi = 4sin^2\left(h/_2\right),\ \varphi = \frac{h^2\left(e^{\frac{-h}{\propto}} - 1\right)}{\frac{-h}{\propto}}$ (5)

for $\propto > 0$

$\psi = \propto \sin(h)$ or $\sin(\propto h)$ $\propto > 0$, $\psi = \frac{(e^{\lambda h} - 1)}{\lambda}$, $\lambda \in R$ (6)

$\beta = cos(h)$ (7)

Moreover grid point calculations will be approximated non-locally

Numerical Experiments

Example 1

$y''\ y' = 4x,\ y(1) = 5\ \ y'(1) = 2$ (8)

Zill and Cullen (2005) gives the analytic solution as

$y = x^2 + 4$ (9)

Scheme1A

Using transformation equations (1 - 4) in equation (8), we have

$\frac{y_{k+2} - 2y_{k+1} + y_k}{\varphi}\left[\frac{(y_{k+1} - y_k)}{\psi}\right] = 4x$ (10)

$\frac{y_{k+1} - 2y_k + y_{k-1}}{\varphi}\left[\frac{(y_k - y_{k-1})}{\psi}\right] = 4x$ (11)

$y_{k+1} = 2y_k - y_{k-1} + \left[\frac{4\varphi\psi x}{(y_k - y_{k-1})}\right]$ (12)

$\varphi = 4sin^2\left(h/_2\right)$, Equation (11) now becomes $\frac{(e^{\lambda h} - 1)}{\lambda}$, for $\lambda \in R$

Scheme1B

Changing the denominator function ψ in (12), we have

$y_{k+1} = 2y_k - y_{k-1} + \left[\frac{4\varphi\psi x}{(y_k - y_{k-1})}\right]$ (13)

$\varphi = 4sin^2\left(h/_2\right)$, $\psi = \sin(\propto h)$, $\propto \in R$

Scheme2A

Adding the control function β to (11) as in equation (4), we have

$\frac{y_{k+1} - 2y_k + y_{k-1}}{\varphi}\left[\frac{(y_k - \beta y_{k-1})}{\psi}\right] = 4x$ (14)

$y_{k+1} = 2y_k - y_{k-1} + \left[\frac{4\varphi\psi x}{(y_k - \beta y_{k-1})}\right]$ (15)

$\varphi = 4sin^2\left(h/_2\right)$, $\psi = \frac{(e^{\lambda h} - 1)}{\lambda}$, $\lambda \in R$, $\beta = cos(h)$

Scheme2B

Changing the denominator function ψ in (15), we have

$$y_{k+1} = 2y_k - y_{k-1} + \left[\frac{4\varphi\psi x}{(y_k - \beta y_{k-1})}\right] \qquad (16)$$

$\varphi = 4\sin^2(h/2)$, $\psi = \sin(\propto h)$, $\propto \in R$, $\beta = \cos(h)$

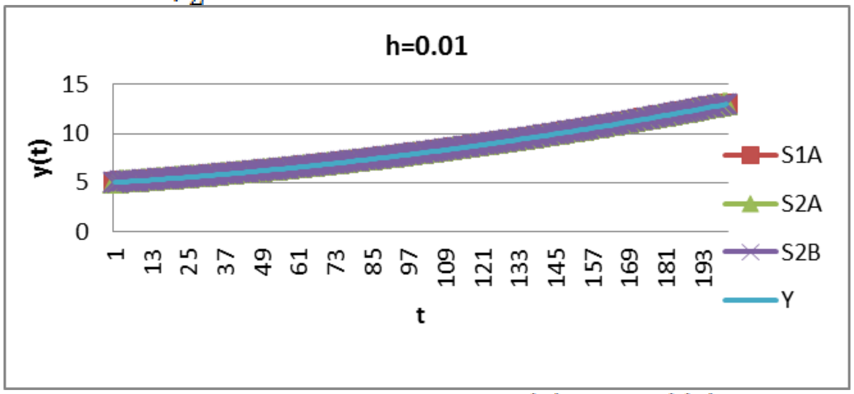

Fig 1: The schemes of $y''\ y' = 4x$, $y(1) = 5$, $y'(1) = 2$, h = 0.01

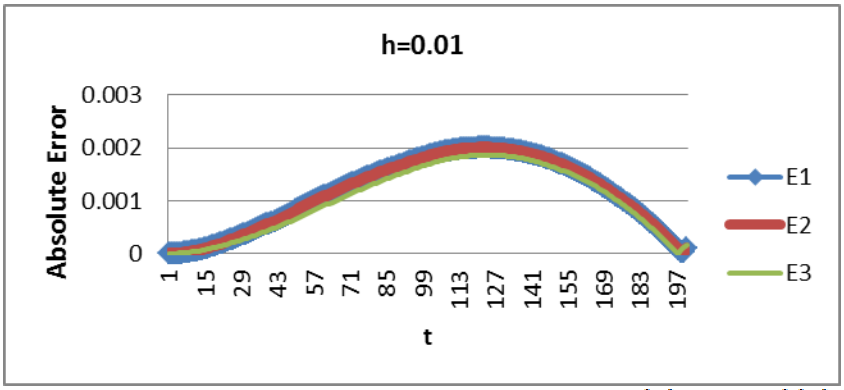

Fig 2: Absolute Error curve for $y''\ y' = 4x$, $y(1) = 5$, $y'(1) = 2$

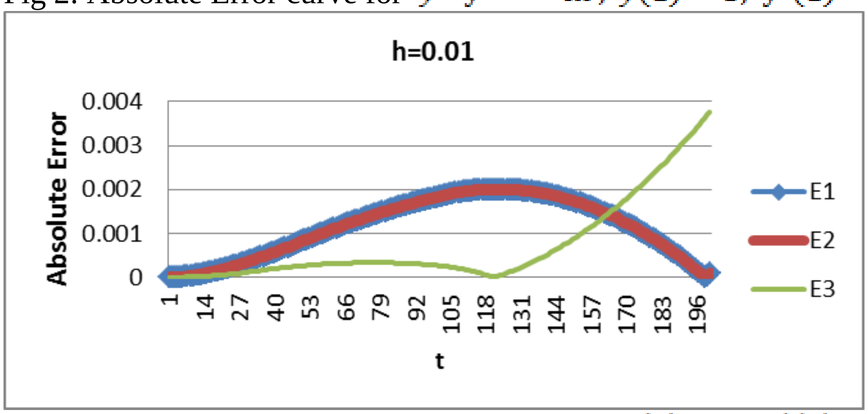

Fig 3: Absolute Error curve for $y''\ y' = 4x$, $y(1) = 5$, $y'(1) = 2$

In this schemes $\propto = 1$, none of the parameters was manipulated for S2B

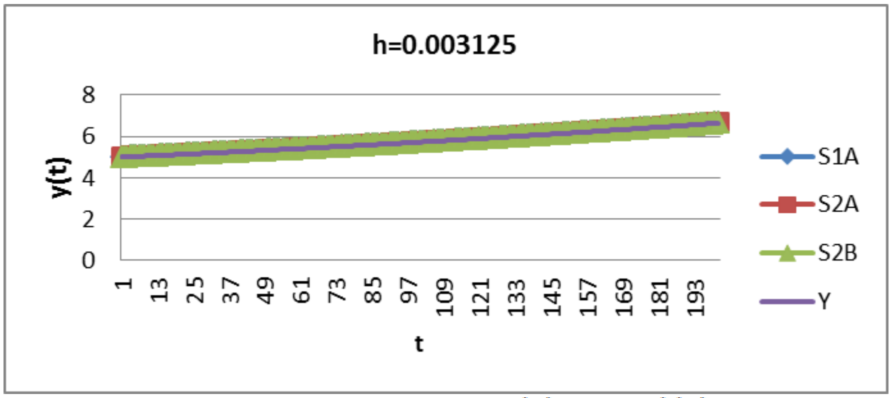

Fig 4: The schemes of $y'' \; y' \; = 4x, \; y(1) = 5, \; y'(1) = 2$

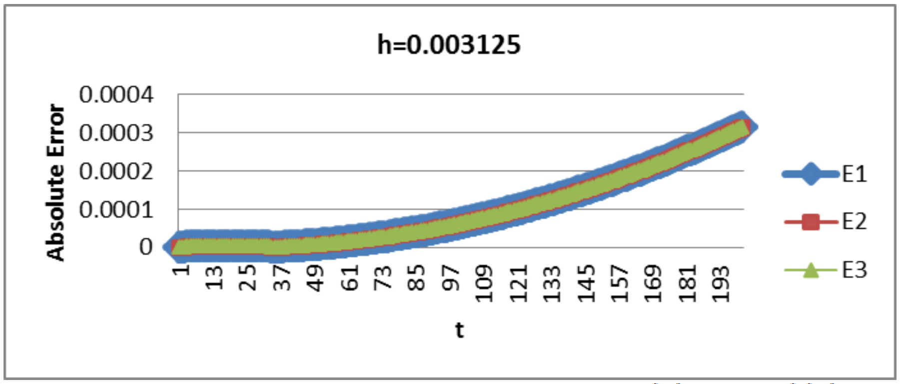

Fig 5: Absolute Error curve for $y'' \; y' \; = 4x, \; y(1) = 5, \; y'(1) = 2$

Example 2

$$y'' - x\,(y')^2 = 0 \; y(0) = 1 \; y'(0) = \tfrac{1}{2} \qquad (17)$$

Zill and Cullen (2005) gives the analytic solution as $y = 1 + \frac{1}{2}\ln\left[\frac{2+x}{2-x}\right]$ (18)

Scheme1A

Using transformation equations (1 - 4) equation in (17), we have

$$\frac{y_{k+2}-2y_{k+1}+y_k}{\varphi} = x\left[\frac{(y_{k+1}-y_k)}{\psi}\right]^2 \qquad (19)$$

$$\frac{y_{k+1}-2y_k+y_{k-1}}{\varphi} = x\left[\frac{(y_k-y_{k-1})}{\psi}\right]^2 \qquad (20)$$

$$y_{k+1} = 2y_k - y_{k-1} + \frac{\varphi x}{\psi^2}(y_k - y_{k-1})^2 \qquad (21)$$

$\varphi = 4\sin^2(h/2)$, $\psi = \frac{(e^{\lambda h}-1)}{\lambda}$, $\lambda \in R$.

Scheme1B

Changing the denominator function ψ in (21), we have

$$y_{k+1} = 2y_k - y_{k-1} + \frac{\varphi x}{\psi^2}(y_k - y_{k-1})^2 \qquad (22)$$

$\varphi = 4\sin^2(h/2)$, $\psi = \sin(\propto h)$, $\propto \in R$

Scheme2A

Adding the control function β to (19) as in equation (3), we have

$$\frac{y_{k+2}-2y_{k+1}+y_k}{\varphi} = x\left[\frac{(y_{k+1}-\beta y_k)}{\psi}\right]^2 \qquad (23)$$

$$\frac{y_{k+1}-2y_k+y_{k-1}}{\varphi} = x\left[\frac{(y_k-\beta y_{k-1})}{\psi}\right]^2 \qquad (24)$$

$$y_{k+1} = 2y_k - y_{k-1} + \frac{\varphi x}{\psi^2}(y_k - \beta y_{k-1})^2 \qquad (25)$$

$\varphi = 4\sin^2(h/2),\ \psi = \frac{(e^{\lambda h}-1)}{\lambda},\ \lambda \in \mathrm{R},\ \beta = \cos(h)$

Scheme2B

Changing the denominator function ψ in (25), we have

$$y_{k+1} = 2y_k - y_{k-1} + \frac{\varphi x}{\psi^2}(y_k - \beta y_{k-1})^2 \qquad (26)$$

$\varphi = 4\sin^2(h/2),\ \psi = \sin(\propto h),\ \propto \in \mathrm{R},\ \beta = \cos(h)$

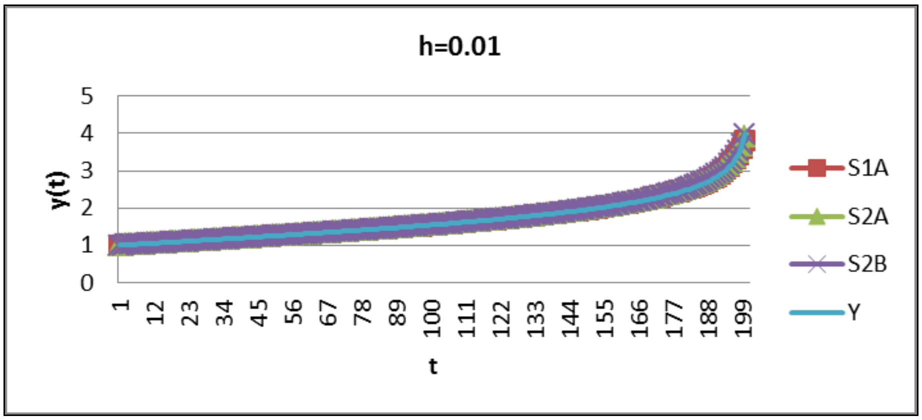

Fig 6: The solution curves of the schemes of $y'' - x\,(y')^2 = 0, y(0) = 1, y'(0) = \frac{1}{2}$

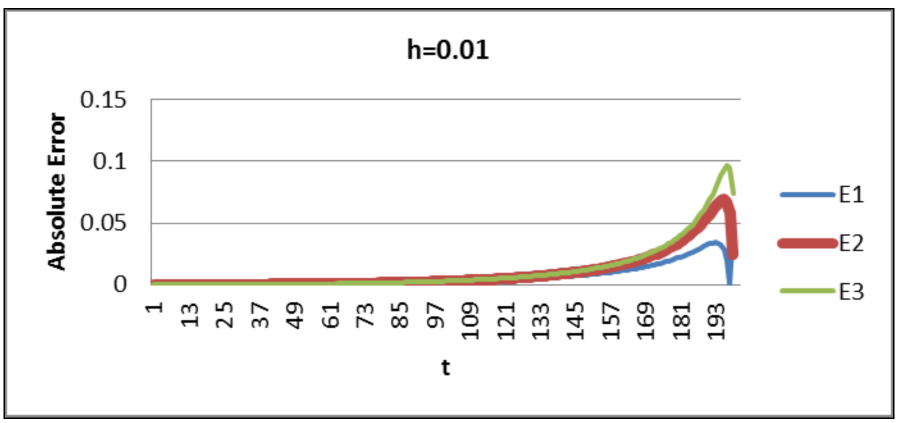

Fig 7: Absolute Error function for the schemes of $y'' - x\,(y')^2 = 0, y(0) = 1, y'(0) = \frac{1}{2}$

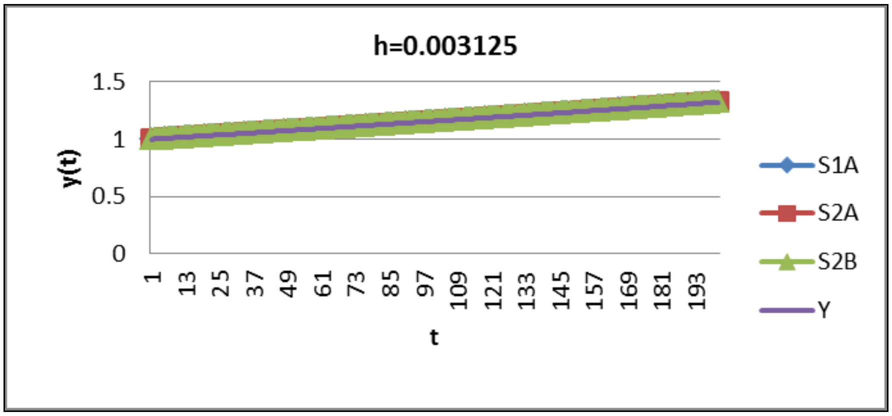

Fig 8: The solution curves of the schemes of $y'' - x\,(y')^2 = 0, y(0) = 1, y'(0) = \frac{1}{2}$

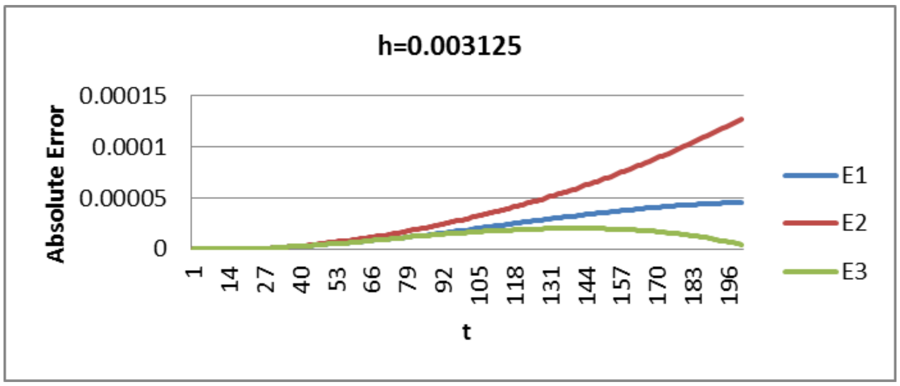

Fig 9: Absolute Error function for the schemes of $y'' - x\,(y')^2 = 0, y(0) = 1, y'(0) = \frac{1}{2}$

Example 3

$$y'' - 2(y' - y) = 0 \;\; y(0) = 1 \;\; y'(0) = \frac{1}{2} \tag{27}$$

Zill and Cullen (2005) gives the analytic solution as $y = \cos(x)e^{x-\pi}$ (28)

Scheme1A

Using transformation equations (1 - 4) in equation (27), we have

$$\frac{y_{k+2} - 2y_{k+1} + y_k}{\varphi} = 2\left[\frac{(y_{k+1} - y_k)}{\psi}\right] - 2y_k \tag{29}$$

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{\varphi} = 2\left[\frac{(y_k - y_{k-1})}{\psi}\right] - 2y_{k-1} \tag{30}$$

$$y_{k+1} = \left(\frac{2\varphi}{\psi} + 2\right)y_k - \left(1 + \frac{2\varphi}{\psi} + 2\varphi\right)y_{k-1} \tag{31}$$

$\varphi = 4\sin^2\left({}^{h}/_{2}\right)$, $\psi = \frac{(e^{\lambda h} - 1)}{\lambda}$, $\lambda \in \mathrm{R}$.

Scheme1B

Changing the denominator function ψ in (31), we have

$$y_{k+1} = \left(\frac{2\varphi}{\psi} + 2\right)y_k - \left(1 + \frac{2\varphi}{\psi} + 2\varphi\right)y_{k-1} \tag{32}$$

$\varphi = 4sin^2(h/_2)$,ψ =sin($\propto h$), $\propto \in$ R .

Scheme2A

$$\frac{y_{k+1}-2y_k+y_{k-1}}{\varphi} = 2\left[\frac{(y_k-\beta y_{k-1})}{\psi}\right] - 2y_{k-1} \quad (33)$$

Adding the control function β to (30) as in equation (4), we have

$$y_{k+1} = \left(\frac{2\varphi}{\psi}+2\right)y_k - \left(1+\frac{2\varphi\beta}{\psi}+2\varphi\right)y_{k-1} \quad (34)$$

$\varphi = 4sin^2(h/_2)$,$\psi = \frac{(e^{\lambda h}-1)}{\lambda}$,λ ∈ R , $\beta = cos(h)$

Scheme2B

Changing the denominator function ψ in (34), we have

$$y_{k+1} = \left(\frac{2\varphi}{\psi}+2\right)y_k - \left(1+\frac{2\varphi\beta}{\psi}+2\varphi\right)y_{k-1} \quad (35)$$

$\varphi = 4sin^2(h/_2)$,ψ =sin($\propto h$), $\propto \in$ R , $\beta = cos(h)$

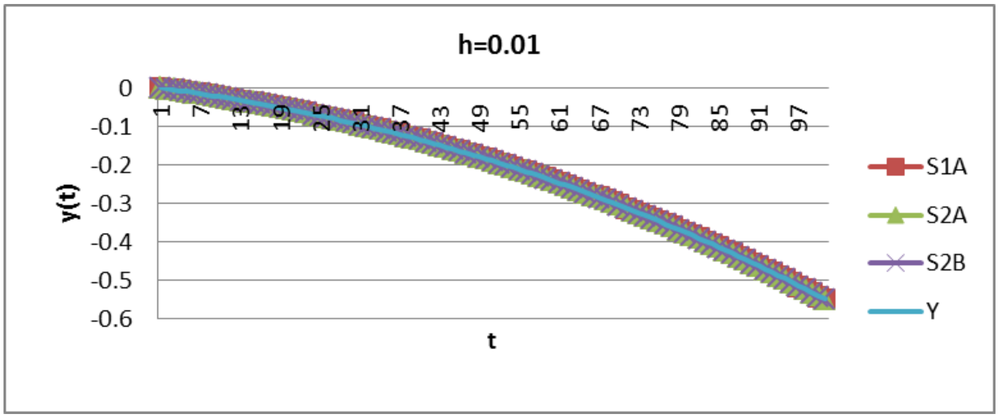

Fig 10: Solution curves for the scheme of $y'' - 2(y' - y) = 0, y(0) = 1, y'(0) = \frac{1}{2}$

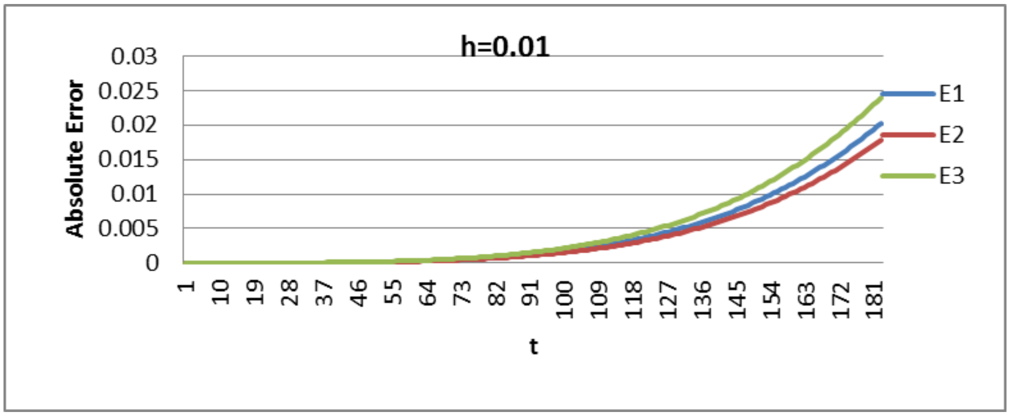

Fig 11: Absolute error function of the schemes of $y'' - 2(y' - y) = 0, y(0) = 1, y'(0) = \frac{1}{2}$

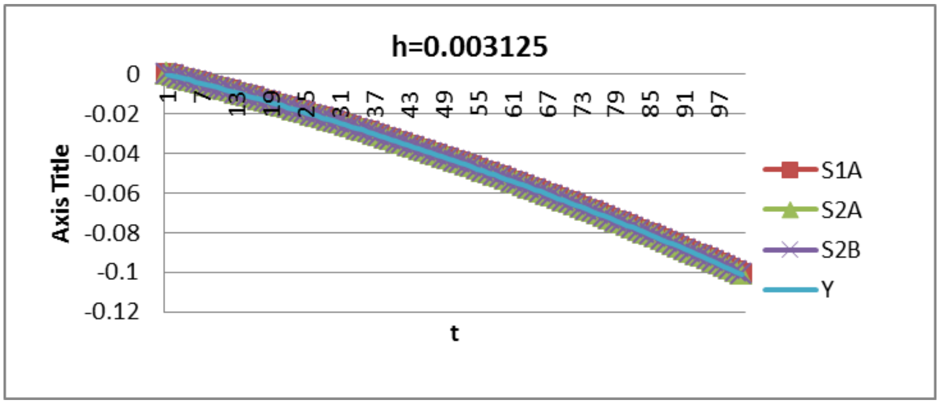

Fig 12: Solution curves for the scheme of $y'' - 2(y' - y) = 0, y(0) = 1, y'(0) = \frac{1}{2}$

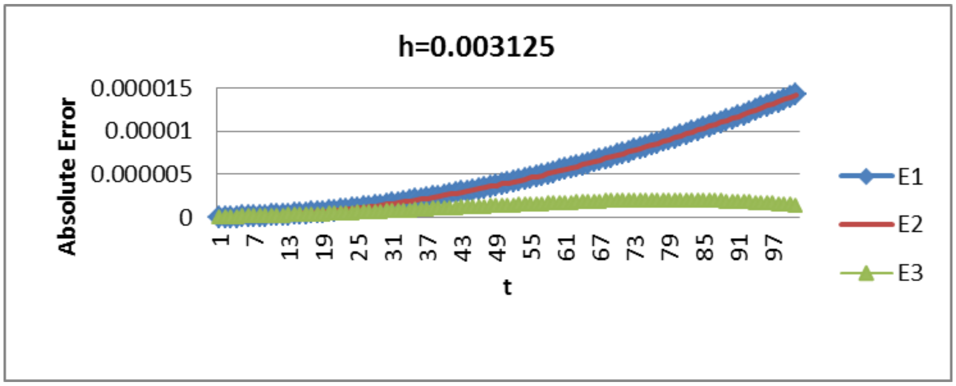

Fig13: Absolute error function of the schemes of $y'' - 2(y' - y) = 0, y(0) = 1, y'(0) = \frac{1}{2}$

Conclusions and Discussion of Results

All the schemes with subscript A (S1A, S2A) have exponential denominator functions while all those with subscript B (S1B, S2B) have trigonometry (sine) denominator functions. It will also be noted that all the second schemes S2A, S2B have control functions cos(h). Control functions are intended to vary the discretization function. It is a means of normalizing and controlling the discrete derivatives. Similar method has been introduced in Mickens (1994)

The numerical experiment has shown that all the derived schemes have the same monotony with the analytic solution see figures 1, 3, 6, 8, 10 and12. The solution curves followed the same pattern. The error functions also showed that all different schemes produces different errors .The errors of the schemes with exponential denominator functions followed the same pattern whether discrete derivatives are controlled or not with see figures 2, 4, 7, 9, 11 and 13. The scheme with denominator (sin (h)) behave erratically if the derivatives are not controlled, this is very consistent with our earlier results in Obayomi and Olabode (2014)

In this paper, we have presented the results for two values of h (0.01 and 0.003125), we can easily see that the schemes behave in the same pattern for $0 < h < 1$. We therefore conclude from the numerical experiments that these schemes are suitable for the simulation of second order ordinary differential equations as proposed.

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